Modular Conjugation and the Implementation of Supersymmetry

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Abstract

Any $\mathbb{Z}_2$-graded C^*-dynamical system with a self-adjoint graded-KMS functional on it can be represented (canonically) as a $\mathbb{Z}_2$-graded algebra of bounded operators on a $\mathbb{Z}_2$-graded Hilbert space, so that the grading of the latter is compatible with the functional. The modular conjugation operator plays a crucial role in this reconstruction. The results are generalized to the case of an unbounded graded-KMS functional having as dense domain the union of a net of C^*-subalgebras. It is shown that the modulus of such an unbounded graded-KMS functional is KMS.

1 Introduction

The notion of a super-KMS functional [8] and more generally a graded-KMS functional [10, 2] have been introduced mainly in connection with the observation of their importance to cyclic cohomology theory [9, 3, 10]. As advocated in [11], the super-KMS functionals seem to be the appropriate substitute for elliptic operators when one passes from finite-dimensional to infinite-dimensional and (or) from commutative to noncommutative geometry, in the sense of [4].

A prototype for a graded-KMS functional on the algebra of bounded operators on a $\mathbb{Z}_2$-graded Hilbert space is given by a regularized supertrace

$$\omega(\,\cdot\,) = \mathrm{str}(\,\cdot\,\rho)$$

with ρ — an even positive trace-class operator. Such functionals appear in finite-volume supersymmetric quantum field theories in thermal background and have been used to define the Witten index [16, 12].

In an abstract C^*-dynamical setting the graded-KMS condition is a natural supersymmetric generalization of the KMS (Kubo, Martin, Schwinger) condition. It is defined with respect to a grading γ of the C^*-algebra $\mathcal{A}$ and a continuous one-parameter *-automorphism group α_t, commuting with γ. Namely ω is a *graded-KMS functional on* $\mathcal{A}$ if it satisfies

$$\omega(ab) = \omega(b^\gamma\,\alpha_i(a)) \tag{1}$$

for any analytic with respect to α_t element $a \in \mathcal{A}$ and any $b \in \mathcal{A}$. The grading γ acts as $b \to b^\gamma := b_+ - b_-$, where $b_\pm$ are the even and odd parts of b respectively and $\alpha_i(a)$ is the value of $\alpha_z(a)$ at $z = \sqrt{-1}$.

As shown in [15], condition (1) arises naturally in the case when ω is a faithful normal (nonpositive) self-adjoint functional on a von Neumann algebra $\mathcal{A}$. More precisely there

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exist a canonical σ-weakly continuous one-parameter group α_t (the "modular group") and a canonical $\mathbb{Z}_2$ grading γ on $\mathcal{A}$, commuting with the automorphism group, and ω is graded-KMS with respect to them. Furthermore, in complete analogy with the standard Tomita-Takesaki theory, (where ω is assumed positive instead of just self-adjoint,) the canonical automorphism group and grading are the unique ones (with certain properties) with respect to which ω is graded-KMS.

A more general notion was studied in [2] — that of a twisted-KMS functional. The defining relation (1) is the same but γ is now an arbitrary *-automorphism, not necessarily an involution. It was shown in [2] that some of the results proven for von Neumann algebras in [15] remain valid in the more general C^* setting and when "graded" is replaced by "twisted". In particular, if ω is twisted-KMS, its modulus, which is a positive functional, is KMS (when normalized to one).

The regularized supertrace, as a prototype for a graded-KMS functional, has two additional properties. It is self-adjoint, i.e., it is real on self-adjoint elements of $\mathcal{A}$. In addition, the grading of the Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is *compatible* with the functional ω. By the latter we mean that for any projection $e \in \mathcal{A}''$ (the weak closure or double commutant of $\mathcal{A}$), $\omega(e) \geq 0$ if $\operatorname{ran} e \subseteq \mathcal{H}_+$ and $\omega(e) \leq 0$ if $\operatorname{ran} e \subseteq \mathcal{H}_-$.

It is interesting to find out the extent to which the regularized supertrace is a generic example of a graded-KMS functional. The present paper tries to answer this. In other words, we study the following question: Given an abstract $\mathbb{Z}_2$-graded C^*-algebra $\mathcal{A}$ with an action of a continuous one-parameter *-automorphism group α_t, preserving the grading and a self-adjoint graded-KMS functional ω, can we represent $\mathcal{A}$ as a $\mathbb{Z}_2$-graded algebra of operators on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$, so that the grading of $\mathcal{H}$ is compatible with ω and the grading of $\mathcal{A}$ is induced from that of $\mathcal{H}$? Further, is ω a regularized supertrace in that representation? Another way is to say that we would like to reconstruct an abstract $\mathbb{Z}_2$-graded C^*-dynamical system with a graded-KMS functional on it and we notice that the grading of the Hilbert space is encoded in the functional, not in the grading of the algebra and in any "good" representation the grading of the algebra should be the one induced from that of the Hilbert space.

We believe that the reconstruction result of this paper has relevance to the transition to infinite volume limit in supersymmetric finite volume quantum field theories. The principal difficulties that one has to overcome are two — reconstruction of the Hilbert space with a nonpositive functional ω and finding a $\mathbb{Z}_2$ graded representation with respect to a $\mathbb{Z}_2$ grading of $\mathcal{H}$, compatible with ω. The first issue is addressed easily by replacing ω with the (canonically defined) positive functional $|\omega|$, called the modulus of ω. The GNS construction with $|\omega|$ and some of its properties will be discussed in the next section. It turns out that this standard GNS representation π has a flaw. It is not a $\mathbb{Z}_2$ graded representation with respect to a $\mathbb{Z}_2$ grading of $\mathcal{H}$, compatible with ω. In fact the whole algebra is mapped onto even operators on $\mathcal{H}$.

Somewhat miraculously, there exists an antiunitary operator J on the GNS Hilbert space with just the right properties, so that if we form a unitary $U := K\,J$, where K is complex conjugation in some basis, then $\pi' := U\,\pi\,U^*$ is a proper $\mathbb{Z}_2$ graded representation. Even though J is defined directly in our work, without any reference to Tomita-Takesaki's theory, it coincides with the modular conjugation operator in the case, when ω is faithful. These developments are the subject of Section 3.

Section 4 generalizes the reconstruction results to the case when ω is unbounded graded-KMS functional with a dense domain in $\mathcal{A}$. The need to consider this more general case stems from the results of, and has been suggested by, Buchholz, Longo and others [2, 1], who showed that the graded-KMS condition for bounded functionals is incompatible with the existence of an automorphism group α_t acting in an asymptotically abelian way, which is the typical situation of a local quantum theory in thermal background in infinite volume.

Motivated by the physical context, we take the domain of ω to be $\mathcal{A}_{\text{loc}} := \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$, the algebra of local "observables", and the algebra $\mathcal{A}$ to be the completion of $\mathcal{A}_{\text{loc}}$. Here $\mathcal{A}(\mathcal{O})$ is an increasing net of C^* subalgebras. We show that the modulus $|\omega|$ of an unbounded graded-KMS functional ω is a positive unbounded KMS functional, thus generalizing results of [15, 2].

2 The GNS construction

In this section we put together mostly known facts and prepare the ground for our main result in Section 3. Let $\mathcal{A}$ be a unital C^*-algebra and ω — a self-adjoint continuous linear functional on it. We list some definitions and facts we shall need.

Two positive functionals φ and ψ are called orthogonal (denoted $\varphi \perp \psi$) if the following equality holds:

$$\|\varphi - \psi\| = \|\varphi\| + \|\psi\| .$$

Every self-adjoint functional ω on a C^*-algebra has a unique decomposition into two orthogonal positive functionals $\omega_\pm$, called *Jordan decomposition* ([13], Sec. 3.2):

$$\omega = \omega_+ - \omega_- , \quad \omega_+ \perp \omega_- . \tag{2}$$

The Jordan decomposition of a self-adjoint functional ω is preserved by any *-automorphism that leaves ω invariant, i.e.

$$\omega \circ \alpha = \omega \quad \Longleftrightarrow \quad \omega_\pm \circ \alpha = \omega_\pm .$$

This follows easily from the fact that *-automorphisms preserve positivity and mutual orthogonality of functionals and from the uniqueness of the Jordan decomposition.

One can associate a (unique) positive functional $|\omega|$ to the self-adjoint ω:

$$|\omega| := \omega_+ + \omega_- . \tag{3}$$

The positive functional $|\omega|$ is called *the modulus* of ω and can be defined in fact for an arbitrary ω as the unique positive functional, satisfying (see [5], Sec. 12.2.9.):

$$\| \, |\omega| \, \| = \|\omega\| , \quad |\omega(a)| \leq \|\omega\| \, |\omega|(a^* a) , \quad a \in \mathcal{A} . \tag{4}$$

An easy exercise shows that the functional, defined by (3) satisfies the conditions in (4).

Consider now the Gelfand - Naimark - Segal (GNS) construction $(\pi, \mathcal{H}, \Omega)$ associated with the positive functional $|\omega|$. Any element $a \in \mathcal{A}$ is represented by a bounded operator $\pi(a)$ on the Hilbert space $\mathcal{H}$ and we have $|\omega|(a) = (\Omega, \pi(a)\,\Omega)$. The algebra $\pi(\mathcal{A})$ is a *-homomorphic image of $\mathcal{A}$ which is in general not isomorpic to $\mathcal{A}$, since $|\omega|$ is not necessarily faithful.

The positive functionals ω_+ and ω_- are dominated by $|\omega|$ and therefore (see [13], Sec. 3.3.) there are unique elements $p_\pm \in \pi(\mathcal{A})'$ with $0 \leq p_\pm \leq 1$, such that for all $a \in \mathcal{A}$

$$\omega_\pm(a) = (\Omega, \pi(a)\, p_\pm \, \Omega) . \tag{5}$$

Since $\omega_1 + \omega_2 = |\omega|$, we have $p_+ + p_- = 1$ and since $\omega_+ \perp \omega_-$ the elements $p_\pm$ are actually mutually orthogonal projections.

The latter statement is obvious in the commutative case, when the Jordan decomposition of a self-adjoint functional reduces to the usual Jordan decomposition of a signed measure and the elements $p_\pm$ are the characteristic functions of the sets on which the measure is positive and negative, respectively. In the general case one way to prove that $p_\pm$ are projections is as follows:

Consider the spectrum σ of the operator p_+. We have $\sigma \subset [0,1]$. Suppose that we can find $\delta > 0$ so that the spectral projection p_0, corresponding to the interval $[\delta, 1/2]$ is not zero. Note that since $p_- = 1 - p_+$, we have $p_- p_0 \geq p_+ p_0$. It is known ([13], Sec. 3.2.) that two positive functionals ω_+ and ω_- are orthogonal if and only if for every $\epsilon > 0$ there is a positive element z in the unit ball of $\mathcal{A}$, such that $\omega_+(1-z) < \epsilon$ and $\omega_-(z) < \epsilon$. We have

$$(\Omega, (1-z)\, p_+\, p_o\, \Omega) \leq (\Omega, (1-z)\, p_+\, \Omega) = \omega_+(1-z) < \epsilon$$

and therefore

$$(\Omega, z\, p_+\, p_0\, \Omega) > (\Omega, p_+\, p_0\, \Omega) - \epsilon \; .$$

But then

$$\omega_-(z) \geq (\Omega, z\, p_-\, p_0\, \Omega) \geq (\Omega, z\, p_+\, p_0\, \Omega) > (\Omega, p_+\, p_0\, \Omega) - \epsilon \; ,$$

which contradicts the statement that $\omega_-(z)$ can be made arbitrarily small. The contradiction shows that $\sigma \cap [\delta, 1/2] = \emptyset$. Similarly, by interchanging the roles of ω_+ and ω_- we show that $\sigma \cap [1/2, 1-\delta] = \emptyset$. Therefore $\sigma = \{0, 1\}$ and p_+ and p_- are projections.

There is an obvious $\mathbb{Z}_2$-grading (orthogonal decomposition) of the Hilbert space — $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with $\mathcal{H}_\pm := p_\pm\, \mathcal{H}$. The functional ω is expressed as a "graded vacuum expectation value":

$$\omega(a) = (\Omega, \pi(a)\, (p_+ - p_-)\, \Omega) = (\Omega, \pi(a)\, \Gamma\, \Omega) \; , \tag{6}$$

where the grading operator Γ has the bock-diagonal form

$$\Gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \; .$$

This grading of $\mathcal{H}$ is obviously compatible with ω. Since every $\pi(a)$ commutes with Γ, the representation is reducible and all $\pi(a)$ have block-diagonal form, i.e., they are all even operators relative to this grading. Thus the GNS representation $(\pi, \mathcal{H}, \Omega)$ is not a $\mathbb{Z}_2$-graded representation of $\mathcal{A}$

Note One may be misled to think that equation (6), relating ω to $|\omega|$ represents the polar decomposition [14] of a self-adjoint functional, but this is wrong. The polar decomposition (of a normal functional on a von Neumann algebra) relates a functional to its modulus via an element of the algebra, while the operator Γ is in the commutant of $\pi(\mathcal{A})$. In the C^* setting the polar decomposition of ω involves an element g of the weak closure of $\pi(\mathcal{A})$, (i.e. its double commutant) as discussed in [2] and further in our paper. This element g does not commute with $\pi(\mathcal{A})$ unless $\mathcal{A}$ is trivially graded. In fact g implements the grading automorphism.

With a slight abuse of notations we will write $|\omega|$, ω and $\omega_\pm$ for the extensions of the respective functionals, using their GNS representations, to the von Neumann algebra $\mathcal{B} := \pi(\mathcal{A})''$ (the weak closure of $\pi(\mathcal{A})$). Thus we have

$$|\omega|(a) := (\Omega, a\, \Omega) \; , \quad \omega(a) := (\Omega, a\, \Gamma\, \Omega) \; , \quad \omega_\pm(a) := (\Omega, a\, p_\pm\, \Omega) \; , \quad a \in \mathcal{B} \; . \tag{7}$$

It is trivial that all four functionals are normal (or σ-weakly continuous), that ω is self-adjoint with $|\omega|$ being its modulus and $\omega_\pm$ giving its Jordan decomposition. In general, the Jordan decomposition $\omega = \omega_+ - \omega_-$ of a normal self-adjoint functional has the following additional properties [6]:

(i) $\omega_\pm$ are normal positive functionals with mutually singular supports, i.e., any $a \in \mathcal{B}$, $a \geq 0$ can be represented (nonuniquely in general) as a sum $a = a_+ + a_-$ so that $\omega_-(a_+) = \omega_+(a_-) = 0$.

(ii) There exist projections $\chi_\pm \in \mathcal{B}$ (not necessarily unique) onto the supports of $\omega_\pm$ with the following properties:

$$\chi_+\chi_- = \chi_-\chi_+ = 0\,, \tag{8}$$

$$\omega_+(a) = \omega(a\chi_+)\,, \qquad a \in \mathcal{B}\,, \tag{9}$$

$$\omega_-(a) = -\omega(a\chi_-)\,, \qquad a \in \mathcal{B}\,. \tag{10}$$

Using these projections and writing $g := \chi_+ - \chi_-$ one can link the modulus $|\omega|$ to ω:

$$|\omega| \,=\, \omega_+ + \omega_- \,=\, g \circ \omega\,. \tag{11}$$

where $g \circ \omega$ is just a notation for the functional defined by $g \circ \omega(a) := \omega(a\,g)$. We also have quite easily:

$$\omega = g \circ |\omega|\,. \tag{12}$$

Formulae (11) and (12) are indeed a special case of the polar decomposition [14] of a normal linear functional.

The projections $\chi_\pm$ are not unique if and only if $|\omega|$ is not faithful. In the case we have at hand $|\omega| = (\Omega, \cdot\, \Omega)$ is faithful on $\pi(\mathcal{A})$ (even though $|\omega|$ may not be faithful as a functional on $\mathcal{A}$). By construction Ω is cyclic for $\pi(\mathcal{A})$ and the existence (see next section) of the antiunitary operator J (modular conjugation) a posteriori shows that Ω is also cyclic for $J\pi(\mathcal{A})J = \pi(\mathcal{A})'$. This implies that Ω is separating for $\pi(\mathcal{A})'' = \mathcal{B}$ which is the same as saying that $|\omega|$ is faithful as a functional on $\mathcal{B}$. Thus $\chi_\pm$ are unique and invariant under any *-automorphism α which leaves $\omega_\pm$ invariant. In addition $\chi_+ + \chi_- = 1$ and $g^2 = 1$. (The considerations in the next section remain valid in the case when $|\omega|$ is not faithful, as long as we choose $\chi_\pm$ to be the unique minimal projections, satisfying (9) and (10). Of course in this case $\chi_+ + \chi_- \neq 1$ and $g^2 \neq 1$.)

3 The $\mathbb{Z}_2$-graded representation

It turns out that the graded-KMS property of the functional ω gives a natural way to define a conjugate representation π', which unlike π respects the grading of $\mathcal{A}$.

First note that any *-automorphism of $\mathcal{A}$, preserving $|\omega|$ can be implemented in the GNS Hilbert space by the adjoint action of a unitary operator. In this way the action of this *-automorphism can be extended from $\mathcal{A}$ to $\mathcal{B} = \pi(\mathcal{A})''$. In particular, $\mathcal{B}$ becomes a $\mathbb{Z}_2$-graded von Neumann algebra with a strongly continuous *-automorphism group α_t, preserving the grading.

Using approximation arguments, it is shown in [2] (Lemma 1) that the extension of ω to $\mathcal{B}$ (which we denote by the same letter) is a graded-KMS functional.

We state now some important results for graded-KMS normal self-adjoint functionals. For the proofs see [15] or [2].

Let ω be a graded-KMS normal self-adjoint functional on the $\mathbb{Z}_2$ graded von Neumann algebra $\mathcal{B} = \mathcal{B}_+ \oplus \mathcal{B}_-$ and $\omega_\pm$, $|\omega|$ are defined as in Section 2. Then the following identities hold:

$$\omega_-(b\chi_+ a) = \omega_+(b\chi_- a) = 0, \qquad a \in \mathcal{B}_+,\ b \in \mathcal{B}, \tag{13}$$

$$\omega_+(b\chi_+ a) = \omega_-(b\chi_- a) = 0, \qquad a \in \mathcal{B}_-,\ b \in \mathcal{B}. \tag{14}$$

The functional $|\omega|$ is a KMS functional. If $a \in \mathcal{B}_+$ is in the left kernel of ω_+, then a^* is in the left kernel of ω_+ as well. The same is true for ω_-. If $a \in \mathcal{B}_-$ is in the left kernel of ω_+, then a^* is in the left kernel of ω_- and vice versa.

A functional ϕ is called even with respect to the grading γ if $\phi(a^\gamma) = \phi(a), \quad a \in \mathcal{B}$. It is easy to see that the graded-KMS property, applied to $\omega(1\,a)$, implies that ω is even and therefore, since the action of γ is a *-automorphism, $\omega_\pm$ and $|\omega|$ have to be even too.

A simple calculation shows that $p_\pm\, a\,\Omega = a\,\chi_\pm\,\Omega$ (but $\neq \chi_\pm\, a\,\Omega$) for any $a \in \mathcal{B}$ and therefore

$$\mathcal{H}_\pm \equiv p_\pm\,\mathcal{H} = (\pi(\mathcal{A})\,\chi_\pm\,\Omega)^- \ ,$$

(where $(\cdot)^-$ signifies completion). We shall need a further decomposition of each $\mathcal{H}_\pm$ into orthogonal direct sums. Define the subspaces $\mathcal{H}^0_\pm := (\pi(\mathcal{A}_+)\,\chi_\pm\,\Omega)^-$ and $\mathcal{H}^1_\pm := (\pi(\mathcal{A}_-)\,\chi_\pm\,\Omega)^-$. The subspaces $\mathcal{H}^{0,1}_+ \subset \mathcal{H}_+$ are mutually orthogonal and so are $\mathcal{H}^{0,1}_- \subset \mathcal{H}_-$, i.e. $\mathcal{H}^0_\pm \perp \mathcal{H}^1_\pm$. We show that orthogonality holds for the dense subspaces. Take e.g., $a \in \mathcal{A}_+$ such that $\omega_-(a^*a) = 0$ and $b \in \mathcal{A}_-$ such that $\omega_-(b^*b) = 0$. Then, because ω_+ is even we get:

$$(\pi(a)\,\Omega, \pi(b)\,\Omega) = |\omega|(a^*b) = \omega_+(a^*b) = 0 \ .$$

Definition: Define an operator $J\ : \mathcal{H} \to \mathcal{H}$ by its action on a dense subspace:

$$J\,\pi(a)\,\Omega := \pi(\alpha_{\frac{i}{2}}(a^*))\,\Omega \ , \quad a \ \text{ analytic in } \mathcal{A} \ .$$

Those familiar with the Tomita–Takesaki theory will realize that the operator J coincides with the *modular conjugation operator* in that theory. Recall that one defines an antilinear operator S as the closure of the operator $S_0\,\pi(a)\,\Omega := \pi(a^*)\,\Omega$. Then the polar decomposition of S is given by $S = J\Delta^{\frac{1}{2}}$, where J is shown to be antiunitary and Δ is self-adjoint. (The *modular operator* Δ then is used to construct a one-parameter automorphism group — the *modular group* via the adjoint action of $\Delta^{i\,t}$ and it turns out that the functional $(\Omega, \cdot\,\Omega)$ is KMS with respect to that modular group.)

Our starting point is different however — we have an automorphism group to begin with and this allows us to define the modular conjugation J in a very simple fashion.

Proposition 1. *J is an antiunitary operator, mapping $\mathcal{H}^1_+$ onto $\mathcal{H}^1_-$ and $\mathcal{H}^1_-$ onto $\mathcal{H}^1_+$ and leaving invariant separately $\mathcal{H}^0_\pm$. Furthermore it is equal to its inverse.*

Proof: First we show the last part:

$$J^2\pi(a)\,\Omega = J\pi(\alpha_{\frac{i}{2}}(a^*))\,\Omega = \pi(\alpha_{\frac{i}{2}}(\alpha_{\frac{i}{2}}(a^*))^*)\,\Omega = \pi(\alpha_{\frac{i}{2}}(\alpha_{-\frac{i}{2}}(a))\,\Omega) = \pi(a)\,\Omega.$$

Next, we know from the results stated above that if $\pi(a)\,\Omega \in \mathcal{H}^1_+$, then $\pi(a^*)\,\Omega \in \mathcal{H}^1_-$, while if $\pi(b)\,\Omega \in \mathcal{H}^0_+$ then. $\pi(b^*)\,\Omega \in \mathcal{H}^0_+$ also. But the automorphisms α_z leave $\mathcal{H}_\pm$ invariant (since ω_+ and ω_- are invariant separately), so $\pi(\alpha_{\frac{i}{2}}(a^*))\,\Omega \in \mathcal{H}^1_-$ and $\pi(\alpha_{\frac{i}{2}}(b^*))\,\Omega \in \mathcal{H}^0_+$. J is obviously antilinear since it involves the antilinear operation $a \to a^*$. Finally J is norm preserving. Take any $\pi(a)\,\Omega$. Then we calculate (remembering that $|\omega|$ is KMS):

$$\begin{aligned}(J\,\pi(a)\,\Omega, J\,\pi(a)\,\Omega) &= |\omega|((\alpha_{\frac{i}{2}}(a^*))^*\,\alpha_{\frac{i}{2}}(a^*)) = |\omega|(\alpha_{-\frac{i}{2}}(a)\,\alpha_{\frac{i}{2}}(a^*)) \\ &= |\omega|(a\,\alpha_i(a^*)) = |\omega|(a^*\,a) = (\pi(a)\,\Omega, \pi(a)\,\Omega) \ .\end{aligned}$$

For the scalar product of two elements $\pi(a)\,\Omega$ and $\pi(b)\,\Omega$ we obtain:

$$\begin{aligned}(J\,\pi(a)\,\Omega, J\,\pi(b)\,\Omega) &= |\omega|(a\,\alpha_i(b^*)) = |\omega|(b^*\,a) \\ = |\omega|((a^*\,b)^*) &= \overline{|\omega|(a^*\,b)} = \overline{(\pi(a)\,\Omega, \pi(b)\,\Omega)} \ .\end{aligned}$$

which completes the proof.

For every antiunitary map J there is a noncanonical unitary map U defined as

$$U := K\,J$$

where K is the operator of complex conjugation with respect to some chosen orthonormal basis in the Hilbert space. As we shall see, different choices of bases lead to isomorphic representations. Since $K^2 = I$ one can easily see that $U^* = JK$ and U is unitary.

We observe the following properties of the different restrictions of U and $\pi(a)$:

$$\begin{array}{ll} U \,:\, \mathcal{H}^0_\pm \to \mathcal{H}^0_\pm\ , & U \,:\, \mathcal{H}^1_\pm \to \mathcal{H}^1_\mp\ , \\ \pi(a) \,:\, \mathcal{H}^0_\pm \to \mathcal{H}^0_\pm\ , & \pi(a) \,:\, \mathcal{H}^1_\pm \to \mathcal{H}^1_\pm \quad a \in \mathcal{A}_+\ , \\ \pi(b) \,:\, \mathcal{H}^0_\pm \to \mathcal{H}^1_\pm\ , & \pi(b) \,:\, \mathcal{H}^1_\pm \to \mathcal{H}^0_\pm\ , \quad b \in \mathcal{A}_-\ . \end{array} \tag{15}$$

Definition: For every $a \in \mathcal{A}$ define an operator $\pi'(a) \,:\, \mathcal{H} \to \mathcal{H}$ as follows:

$$\pi'(a) := U\pi(a)U^*\ .$$

We now prove the main result of this section.

Proposition 2. *The operators $\pi'(a)$ are bounded for any $a \in \mathcal{A}$. The map $\pi' : \mathcal{A} \to L(\mathcal{H}_+ \oplus \mathcal{H}_-)$ gives a ($\mathbb{Z}_2$ graded) representation of the $\mathbb{Z}_2$ graded C^*-algebra $\mathcal{A}$.*

Before proceeding with the proof we would like to make the following comments. A representation of a $\mathbb{Z}_2$ graded C^*-algebra $\mathcal{A}$ is by definition a *-algebra homomorphism $\pi' \,:\, \mathcal{A}_+ \oplus \mathcal{A}_- \to (L_+ \oplus L_-)(\mathcal{H}_+ \oplus \mathcal{H}_-)$ ($L(\mathcal{H})$ meaning all bounded operators on $\mathcal{H}$), which commutes with (preserves) the grading. The grading $L_+ \oplus L_-$ is the natural one induced from the grading of $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

Proof:

(i) Algebra homomorphism:

As π' is obviously linear, we only need to show that $\pi'(ab) = \pi'(a)\pi'(b)\,, \forall a, b \in \mathcal{A}$. This is obvious from the definition of π' and the fact that U is unitary.

(ii)π' commutes with the grading:

This is evident from the way π' was constructed. For $a \in \mathcal{A}_+$, $\pi'(a) \,:\, \mathcal{H}_\pm \to \mathcal{H}_\pm$, i.e., $\pi'(a) \in L_+(\mathcal{H}_+ \oplus \mathcal{H}_-)$, and for $b \in \mathcal{A}_-$, $\pi'(b) \,:\, \mathcal{H}_\pm \to \mathcal{H}_\mp$, i.e. $\pi'(b) \in L_-(\mathcal{H}_+ \oplus \mathcal{H}_-)$.

(iii) A *-homomorphism:

This is also immediate from the definition

$$\pi'(a)^* = (U\,\pi(a)\,U^*)^* = U\,\pi(a)^*\,U^* = U\,\pi(a^*)\,U^* = \pi'(a^*)$$

where we used the fact that the standard GNS representation π is a *-homomorphism.

Note One can find mentioned in the literature (see, e.g., [7]) the conjugate-linear representation $\pi_r := J\pi J$ which is not a representation in the usual sense, i.e., it is not an algebra homomorphism. It is well known that conjugating an element $\pi(a)$ with the modular conjugation operator J one gets an element in the commutant $\pi(\mathcal{A})'$. Thus we have $\pi_r(\mathcal{A}) \equiv J\,\pi(\mathcal{A})\,J\ \subset \pi(\mathcal{A})'$. In general we do not have equality since $\pi(\mathcal{A})$ is not a von Neumann algebra.

The next few statements have easy proofs which we omit.

In the representation π' the functional ω is again expressed as a graded vacuum expectation value.

$$\omega(a) = (\Omega, \Gamma\pi'(a)\Omega)\,.$$

The operator g implementing the grading is mapped to Γ, when conjugated with U:

$$U\,g\,U^* \equiv U\,(\chi_+ - \chi_-)\,U^* = \Gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

The vacuum Ω is a cyclic vector for the representation π'.

The next statement treats the question of uniqueness (up to isomorphism) of the representation π'.

Proposition 3. *Let $(\mathcal{H}',\pi',\Omega')$ and $(\mathcal{H}'',\pi'',\Omega'')$ be two graded representations of $\mathcal{A}$ in $\mathbb{Z}_2$ graded Hilbert spaces and let Γ' and Γ'' are the operators represented both as $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ relative to the respective decompositions of $\mathcal{H}'$ and $\mathcal{H}''$. Suppose that the two representations satisfy $(\Omega',\Gamma'\pi'(a)\Omega') = (\Omega'',\Gamma''\pi''(a)\Omega'')$ and $(\Omega',\pi'(a)\Omega') = (\Omega'',\pi''(a)\Omega'')$ $\forall a \in \mathcal{A}$. Then π' and π'' are unitarily equivalent as graded representations with an intertwining map V that respects the gradings of the two Hilbert spaces*

4 The Unbounded Case

As mentioned in the Introduction, there is strong evidence that the framework of bounded graded-KMS functionals may be too restrictive. In particular, it is incompatible with a requirement for locality in infinite-volume quantum field theory [2, 1]. Thus it makes sense to try to generalize the results in the last section to the case of unbounded graded-KMS functionals.

We will assume (see, e.g., [7], Ch. III.) that we are given a *net of C^*-algebras*

$$\mathcal{O} \to \mathcal{A}(\mathcal{O})$$

assigning to each bounded open region in space-time $\mathcal{O}$ a C^*-algebra $\mathcal{A}(\mathcal{O})$ with the property

$$\mathcal{O}_1 \subset \mathcal{O}_2 \quad \Rightarrow \quad \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \ .$$

The *algebra of local "observables"* is defined as

$$\mathcal{A}_{\rm loc} := \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$$

and the algebra $\mathcal{A}$ is the completion of $\mathcal{A}_{\rm loc}$.

Note Strictly speaking, from the point of view of Algebraic Quantum Field Theory, odd operators with respect to the decomposition of the Hilbert space into bosonic and fermionic sectors are not observables.

We assume that $(\mathcal{A},\ \alpha_t,\ \gamma)$ is a $\mathbb{Z}_2$ graded C^*-dynamical system with the grading $*$-automorphism γ preserving each local algebra $\mathcal{A}(\mathcal{O})$ and commuting with the one-parameter $*$-automorphism group α_t. Following (almost exactly) [1], we adopt the following

Definition: The functional ω will be called *unbounded graded-KMS functional* whenever its domain Dom ω is a dense $*$-subalgebra of $\mathcal{A}$, which is γ and α_t invariant and the following conditions are satisfied:

(A) For any two elements $a, b \in \mathcal{A}$, such that $a\,\alpha_t(b) \in \text{Dom}\ \omega$ and $\alpha_t(b)\,a \in \text{Dom}\ \omega$ for all t, there exists a (unique) complex function $F_{a,b}(z)$ defined on the strip $\{z\,|\,0 \le \text{Im}z \le 1\}$ which is analytic in the interior of that strip and satisfies on the boundaries

$$F_{a,b}(t) = \omega(a\,\alpha_t(b)) \tag{16}$$
$$F_{a,b}(t+i) = \omega(\alpha_t(b)\,a^\gamma) \tag{17}$$

(B) For a, b as above we have the following growth condition:

$$|F_{a,b}(t+is)| \le C(1+|t|)^N, \quad 0 \le s \le 1, \tag{18}$$

where $C \in \mathbb{R}_+$ and $N \in \mathbb{N}$ are constants, depending on a and b.

Note In the case of bounded functionals it is known that condition (A) above, together with a requirement that $F_{a,b}$ is bounded on the strip is equivalent to the condition in Equation 1, which we adopted initially as a definition for the graded-KMS property (see, e.g., [13], 8.12.3 for a proof in the nongraded case). The reason for preferring the current definition for the unbounded case is that there is no reason to believe that all analytic elements in $\mathcal{A}$ (in some natural physical context) are in the domain of ω.

We will assume that $\mathcal{A}_{\mathrm{loc}} = \mathrm{Dom}\ \omega$. We do not assume that $\mathcal{A}(\mathcal{O})$ are unital or that the unit of $\mathcal{A}$ is in the domain of ω.

It follows from (A) and (B) that any unbounded graded-KMS functional is α_t-invariant and even, i.e., γ-invariant([1], Proposition 5.3).

In the following considerations we take ω to be self-adjoint. Given an unbounded self-adjoint functional ω, one can define a unique unbounded positive functional $|\omega|$ by requiring the restriction of $|\omega|$ to any $\mathcal{A}(\mathcal{O})$ to be the unique modulus of the restriction of ω to that subalgebra. A simple argument, using the structure of the net of local algebras, shows that the definition is unambiguous. Similarly, the Jordan decomposition $\omega = \omega_+ - \omega_-$ can be defined in this setting.

The automorphism group α_t does not (in general) preserve the local subalgebras $\mathcal{A}(\mathcal{O})$ separately. In fact it usually has the meaning of time translations acting on the local observables and satisfies $\alpha_t(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\mathcal{O}^t)$, where O^t is the time translate of the region $\mathcal{O}$. It does, however, preserve the whole net of local algebras. From this, the uniqueness of the Jordan decomposition and the α_t-invariance of ω it follows that $\omega_\pm$ and $|\omega|$ are α_t-invariant. The grading automorphism γ is an automorphism of every $\mathcal{A}(\mathcal{O})$ separately, so a simpler argument shows that $\omega_\pm$ and $|\omega|$ are even.

Let $L_{|\omega|} := \{a \in \mathcal{A}_{\mathrm{loc}} \mid |\omega|(a^*a) = 0\}$, which is a left ideal in $\mathcal{A}_{\mathrm{loc}}$.The positive functional $|\omega|$ determines an inner product on the space $\mathcal{A}_{\mathrm{loc}}/L_{|\omega|}$ and we define the Hilbert space $\mathcal{H}$ to be the completion of that space. The algebra $\mathcal{A}$ is then implemented as an algebra of bounded operators on that Hilbert space (via left multiplication).We use π to denote this representation. Let $\eta : \mathcal{A}_{\mathrm{loc}} \to \mathcal{H}$ denote the canonical map, projecting an element of $\mathcal{A}_{\mathrm{loc}}$ onto $\mathcal{A}_{\mathrm{loc}}/L_{|\omega|}$, followed by embedding into $\mathcal{H}$. It is important to notice that we implement the whole algebra $\mathcal{A}$ in this way, not just $\mathcal{A}_{\mathrm{loc}}$. Although for $a \in \mathcal{A}$ and $b \in \mathcal{A}_{\mathrm{loc}}$, the product ab is generally not in $\mathcal{A}_{\mathrm{loc}}$, we can still make sense of the element $\pi(a)\,\eta(b) \in \mathcal{H}$ since

$$||\pi(a)\,\eta(b)||^2 = |\omega|(b^*a^*a\,b) \leq ||a||^2|\omega|(b^*b) = ||a||^2\,||\eta(b)||^2 < \infty\ .$$

Strictly speaking the equation above can be given sense by approximating a with elements from $\mathcal{A}_{\mathrm{loc}}$ and passing to the limit.

Each algebra $\mathcal{A}(\mathcal{O})$ possesses an approximate unit u_λ and one can take (see [13], Section 3.3) the element $\Omega_{\mathcal{O}} := \lim_\lambda \eta(u_\lambda)$. We get a net of Hilbert subspaces $\mathcal{H}(\mathcal{O}) := (\pi(\mathcal{A}(\mathcal{O}))\,\Omega_{\mathcal{O}})^- \subset \mathcal{H}$. The restriction of $\pi(\mathcal{A}(\mathcal{O}))$ to $\mathcal{H}(\mathcal{O})$ is a standard GNS representation with a cyclic and separating vector $\Omega_{\mathcal{O}}$. In particular for any $a \in \mathcal{A}(\mathcal{O})$ we have

$$|\omega|(a) = (\Omega_{\mathcal{O}}, \pi(a)\,\Omega_{\mathcal{O}})$$

and there are projections $p_{\mathcal{O}\pm} \in (\pi(\mathcal{A}))'$ as in Section 2, relating (the restrictions of) $\omega_\pm$ and $|\omega|$. It is easy to see that the nets $p_{\mathcal{O}+}$ and $p_{\mathcal{O}-}$ are nondecreasing and therefore there are limits p_+ and p_- in $(\pi(\mathcal{A}))'$ so that for any $a \in \mathcal{A}(\mathcal{O})$ we will have Equations 5 and 6 with Ω replaced by $\Omega_{\mathcal{O}}$. As in Section 2, we take the weak closure $\mathcal{B} := \pi(\mathcal{A})''$ and note that $\mathcal{B}$ inherits a net of local von Neumann subalgebras $\mathcal{B}(\mathcal{O})$. The functionals $|\omega|$, ω and $\omega_\pm$ are extended to $\mathcal{B}_{\mathrm{loc}} := \bigcup \mathcal{B}(\mathcal{O})$ in the obvious way, applying the analogs of Equation 7. Using the results for normal functionals on the local algebras $\mathcal{B}(\mathcal{O})$ we get nets of projections $\chi_{\mathcal{O}+}$ and $\chi_{\mathcal{O}-}$ and their limits χ_+ and χ_-, so that Equations 11 and 12

remain valid in this more general context. It is quite clear that $\chi_{\mathcal{O}+} + \chi_{\mathcal{O}-} = 1_{\mathcal{B}(\mathcal{O})}$ (the unit of $\mathcal{B}(\mathcal{O})$) and by passing to the limit we have $\chi_+ + \chi_- = 1$ and similarly $p_+ + p_- = 1$.

The projections $\chi_\pm$ and $p_\pm$ must be even and α_t-invariant. The former is obvious. The latter is due to the invariance of ω and $\omega_\pm$ and the uniqueness of Sakai's polar decomposition on each $\mathcal{B}(\mathcal{O})$. We should emphasize that each $\chi_{\mathcal{O}+}$ and $\chi_{\mathcal{O}-}$ is not α_t-invariant, but α_t induces automorphisms of the whole nets of projections and the limits $\chi_\pm$ are invariant.

Proposition 4. *Let ω be a self-adjoint unbounded graded-KMS functional in the context of the preceding paragraphs and let $|\omega|$ and $\omega_\pm$ be the modulus and the positive and negative parts of ω, respectively, in that context. Then $|\omega|$ is an unbounded KMS functional with the same domain. (The definition of the latter should be clear — it is the same as the definition for a graded-KMS functionals, without the grading automorphism γ in* Equations 16 and 17.) *If $a \in \mathcal{A}_+$ (i.e., a is an even element) and a is in the left kernel of ω_+, then a^* is in the left kernel of ω_+ as well. The same is true for ω_-. If $a \in \mathcal{A}_-$ (i.e., a is an odd element) and a is in the left kernel of ω_+, then a^* is in the left kernel of ω_- and vice versa.*

Proof: The proof relies on identities, analogous to Equations 13 and 14. To avoid unnecessary complications we will only consider elements $a, b \in \mathcal{A}$. The projections $\chi_\pm$ are not in $\mathcal{A}$ but in $\mathcal{B}$, so Equations 13 and 14 make sense in the GNS-representation described in this section. First we will need to extend ω to a certain class of elements outside of $\mathcal{A}_{\text{loc}}$, which will be analytic with respect to α_t. Recall that an element a is called analytic if $\alpha_t(a)$ has an extension to an entire function $\alpha_z(a)$. For every $a \in \mathcal{A}_{\text{loc}}$ define a family of elements (see [13], 8.12.1)

$$a_{\sigma,z} := (\pi)^{-1/2}\,\sigma^{-1} \int \alpha_t(a)\, \exp(-(t-z)^2/\sigma^2)\, dt\ , \quad \sigma \in \mathbb{R}, \quad z \in \mathbb{C}\ . \tag{19}$$

We have $a_{\sigma,0} \to a$ as $\sigma \to 0$ (norm convergence) and $\alpha_z(a_{\sigma,\zeta}) = a_{\sigma,z+\zeta}$. Notice that the elements $a_{\sigma,z}$ are generally not in $\mathcal{A}_{\text{loc}}$ but can of course be approximated by elements from $\mathcal{A}_{\text{loc}}$, for example by taking Riemann sums over increasing intervals in place of the integral in Equation 19.

We take $\tilde{\mathcal{A}}_{\text{loc}}$ to be the algebra generated by $\mathcal{A}_{\text{loc}}$ and elements of the type $a_{\sigma,z}$ (in the sense of sums of finite products). The functionals ω, $|\omega|$ and $\omega_\pm$ are extended in an obvious way to $\tilde{\mathcal{A}}_{\text{loc}}$. For example for $a, b \in \mathcal{A}_{\text{loc}}$ and $c_{\sigma,z}$, $d_{\sigma',z'}$ as above, we define

$$\omega(a\, c_{\sigma,z} b\, d_{\sigma',z'}) := (\pi\sigma\sigma')^{-1} \iint dt\, dt' \exp(-(t-z)^2/\sigma^2)\, \exp(-(t'-z')^2/\sigma'^2)\, \omega(a\, \alpha_t(c)\, b\, \alpha_{t'}(d))\ ,$$

which makes sense due to the growth condition Equation 18. Furthermore, for every $a_{\sigma,z} \in \tilde{\mathcal{A}}_{\text{loc}}$, taking a sequence $a_n \to a_{\sigma,z}$, $a_n \in \mathcal{A}_{\text{loc}}$, we have

$$\omega_\pm(a_n^* a_n) \geq 0 \quad \Rightarrow \lim_{n\to\infty} \omega_\pm(a_n^* a_n) = \omega_\pm(a_{\sigma,z}^* a_{\sigma,z}) \ \geq 0\ ,$$

which shows that $\omega_\pm$ and similarly $|\omega|$ remain positive on $\tilde{\mathcal{A}}_{\text{loc}}$.

The functional ω is graded-KMS on $\tilde{\mathcal{A}}_{\text{loc}}$. Indeed, for $a \in \mathcal{A}_{\text{loc}}$ and $b' = b_{\sigma,\zeta}$ the function

$$\begin{aligned} F_{a,b'}(t) := \omega(a\, \alpha_t(b')) &= \int \exp(-(t'-\zeta)^2/\sigma^2)\, \omega(a\, \alpha_{t+t'}(b))\, dt' \\ &= \int \exp(-(t'-\zeta)^2/\sigma^2)\, F_{a,b}(t+t')\, dt' \end{aligned}$$

extends to the entire function $F_{a,b'}(z) = \omega(a\, \alpha_z(b')) = \omega(a\, b_{\sigma,z+\zeta})$ and

$$\begin{aligned} F_{a,b'}(t+i) = \omega(a\, \alpha_{t+i}(b')) &= \int \exp(-(t'-\zeta)^2/\sigma^2)\, F_{a,b}(t+t'+i)\, dt' \\ &= \int \exp(-(t'-\zeta)^2/\sigma^2)\, \omega(\alpha_{t+t'}(b)\, a^\gamma)\, dt' = \omega(\alpha_t(b')\, a^\gamma)\ , \end{aligned}$$

where we used the graded-KMS condition Equations 16 and 17 on $\mathcal{A}_{\rm loc}$ in the last line.

The analogs of Equations 13 and 14 for arbitrary $a, b \in \tilde{\mathcal{A}}_{\rm loc}$ are proven as follows. Take, e.g., a to be odd analytic in $\tilde{\mathcal{A}}_{\rm loc}$ and consider for example

$$\begin{aligned} 0 \leq \omega_+(a^*\chi_+ a) &= \omega_+(a^*\chi_+ a\,\chi_+) = \omega(a^*\chi_+ a\,\chi_+) = \omega(\alpha_{-i}(a\,\chi_+)(a^*\chi_+)^\gamma) \\ &= -\omega(\alpha_{-i/2}(a)\,\chi_+\alpha_{i/2}(a^*)\chi_+) = -\omega_+(\alpha_{-i/2}(a)\,\chi_+(\alpha_{-i/2}(a))^*) \leq 0\,. \end{aligned}$$

This inequality shows that $\omega_+(a^*\chi_+ a) = 0$. We have used the fact that both $a^*\chi_+ a$ and $\alpha_{-i/2}(a)\,\chi_+(\alpha_{-i/2}(a))^*$ are positive elements of $\tilde{\mathcal{A}}_{\rm loc}$, the α_t-invariance of ω and χ_+ as well as χ_+ being odd and the property of $*$-conjugation $(\alpha_z(a))^* = \alpha_{\bar{z}}(a^*)$. Now, for arbitrary elements $a, b \in \mathcal{A}_{\rm loc}$, a – odd, we show first that $\omega_+(a^*\chi_+ a) = 0$ by approximating a with analytic elements from $\tilde{\mathcal{A}}_{\rm loc}$ and then, using Schwarz inequality, we get $\omega_+(b\chi_+ a) = 0$

Showing that $|\omega|$ is a KMS functional is already easy. Take, e.g., a – odd and consider the function $G_{b,a}(t) := |\omega|(b\,\alpha_t(a))$. We need only look at the case b – odd, since $|\omega|$ is even functional. We calculate

$$\begin{aligned} |\omega|(b\,\alpha_t(a)) &= \omega_+(b\,\chi_+\,\alpha_t(a)) + \omega_+(b\,\chi_-\,\alpha_t(a)) + \omega_-(b\,\chi_+\,\alpha_t(a)) + \omega_-(b\,\chi_-\,\alpha_t(a)) \\ &= \omega_+(b\,\chi_-\,\alpha_t(a)) - \omega_-(b\,\chi_-\,\alpha_t(a)) - \omega_+(b\,\chi_+\,\alpha_t(a)) + \omega_-(b\,\chi_+\,\alpha_t(a)) \\ &= -\omega(b\,(\chi_+ - \chi_-)\,\alpha_t(a)) = -F_{b(\chi_+-\chi_-),a}(t)\ . \end{aligned}$$

The first and the fourth terms in the right-hand side of the first line are zero, so we have switched the signs in front of them in the next line. We know that there is an analytic extension of $F_{b(\chi_+-\chi_-),a}(t)$ to the strip $\{z\,|\,0 \leq \mathrm{Im} z \leq 1\}$ and thus the same is true for $G_{b,a}(t)$ and

$$G_{b,a}(t+i) = -F_{b(\chi_+-\chi_-),a}(t+i) = -\omega(\alpha_t(a)\,b^\gamma(\chi_+ - \chi_-)) = |\omega|(\alpha_t(a)\,b)\ . \qquad (20)$$

A similar argument demonstrates the KMS property for $|\omega|$ when a and b are even.

The last statement of the proposition is a simple consequence of the analogs of Equations 13 and 14. For example, suppose a is odd and a is in the left kernel of ω_+. This means that $a = a\,\chi_-$ and therefore

$$\omega_-(a\,a^*) = \omega_-(a\,\chi_-\,a^*) = 0\ .$$

The proof is complete.

Finally, the structure can be completed by defining the modular conjugation operator J. For this we first extend by continuity the canonical map $\eta : \mathcal{A}_{\rm loc} \to \mathcal{H}$ to $\tilde{\mathcal{A}}_{\rm loc}$. This is possible, since for any $a \in \tilde{\mathcal{A}}_{\rm loc}$ we saw that $|\omega|(a^*a) < \infty$. For any analytic element $a \in \tilde{\mathcal{A}}_{\rm loc}$ we define

$$J\,\eta(a) := \eta(\alpha_{\frac{i}{2}}(a^*))\ .$$

This defines J on a dense subset of $\mathcal{H}$. The proof that J is antiunitary is the same as before. Taking an operator of complex conjugation K and defining the unitary map $U := KJ$, we define a representation $\pi' := U\pi U^*$.

The orthogonal decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with $\mathcal{H}_\pm := p_\pm\mathcal{H}$ is compatible with the unbounded self-adjoint functional ω. Equations 15 remain valid and it becomes clear that the main result of Section 3 (Proposition 2) now extends to the case of ω being an unbounded graded-KMS functional.

References

[1] D. Buchholz, H. Grundling, Algebraic Supersymmetry: A case study, arXiv:math-ph/0604044 v1 20 Apr 2006

[2] D. Buchholz, R. Longo, Graded KMS Functionals and the Breakdown of Supersymmetry, *Adv.Theor.Math.Phys.* **3** (1999) 615-626; Addendum-ibid. **3** (1999) 1909-1910.

[3] A. Connes, Entire cyclic cohomology of Banach algebras and characters of θ-summable Fredholm modules, *K-Theory* **1** (1988) 519-548.

[4] A. Connes, "Noncommutative Geometry", Academic Press, 1994.

[5] J. Dixmier, "C^*-Algebras", North-Holland, 1977.

[6] J. Dixmier, "Von Neumann algebras", North-Holland, Amsterdam, New York, Oxford, 1981.

[7] R. Haag, "Local Quantum Physics", Springer-Verlag, Berlin, Heidelberg, 1992.

[8] A. Jaffe, A. Lesniewski, K. Osterwalder, Super-KMS functionals and entire cyclic cohomology, *K-Theory* **2**, No 6, (1989), 675 - 682.

[9] A. Jaffe, A. Lesniewski, K. Osterwalder, Quantum K-theory I, *Commun. Math. Phys.* **118** (1988) 1-14.

[10] D. Kastler, Cyclic cocycles from graded KMS functionals, *Commun. Math. Phys.* **121** (1989) 345–350.

[11] D. Kastler, Cyclic cohomology, supersymmetry and KMS states. The KMS states as generalized elliptic operators, Preprint CPT-89/P.2280. May 1989.

[12] R. Longo, Notes for a Quantum Index Theorem, *Commun.Math.Phys.* **222** (2001) 45-96

[13] G. Pedersen, "C^*-algebras and their automorphism groups", Academic press, London, New York, San Francisco (1979).

[14] S. Sakai, On linear functionals of W^*-algebras, *Proc. Japan Acad.* **34** (1958) 571-574.

[15] O. Stoytchev, The modular group and super-KMS functionals, *Lett. Math. Phys.* **27** (1993) 43-50.

[16] E. Witten, Constraints on supersymmetry breaking, *Nucl. Phys.* **B 202** (1982) 253.